

University of Belgrade
Faculty of Mathematics

Homogenous pp-wave metrics on 4-dimensional Lorentz Lie groups

Tijana Šukilović

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The pp-waves

- In general relativity, the *pp-waves* are an important family of exact solutions of Einstein's field equation.
- The pp-waves solutions model radiation moving at the speed of light.
- A **pp-wave** is a space-time that admits a special null direction along which the wave travels, with wavefronts that are planar in a geometric sense.
- A **plane wave** is a pp-wave whose profile is independent of transverse coordinates, i.e. constant across each wavefront.

Plane wave

Roger Penrose “*Any space-time has a plane wave as a limit*”

A corresponding procedure can also be applied to any entire (properly embedded) null geodesic γ in any space-time.

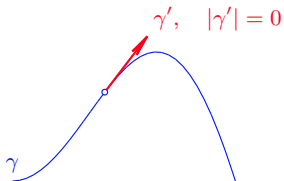
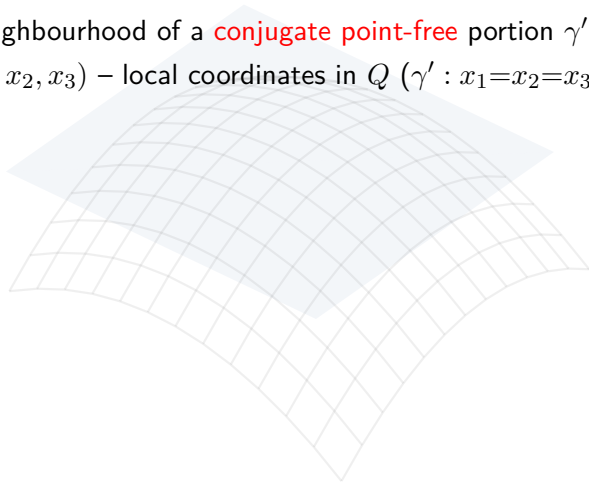


Figure 3: Null geodesic

But whereas T_p is, in an essential way, a flat space, the corresponding procedure applied to γ yields a curved space W_γ known as a **plane wave**.

Mathematical procedure

- Q – neighbourhood of a **conjugate point-free** portion γ' of γ ;
- (x_0, x_1, x_2, x_3) – local coordinates in Q ($\gamma' : x_1=x_2=x_3=0$).



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Possible issue

- If M satisfies the weak energy condition, null geodesic completeness and a (physically reasonable) generality condition, it follows that every null geodesic actually does contain a pair of conjugate points.
- Coordinates (x_0, x_1, x_2, x_3) cannot cover portion of γ containing a pair of conjugate points.
- Thus, for the whole of γ , a system of overlapping coordinate neighbourhoods is required.

Mathematical procedure

- Q – neighbourhood of a conjugate point-free portion γ' of γ ;
- (x_0, x_1, x_2, x_3) – local coordinates in Q ($\gamma' : x_1=x_2=x_3=0$).
- metric:

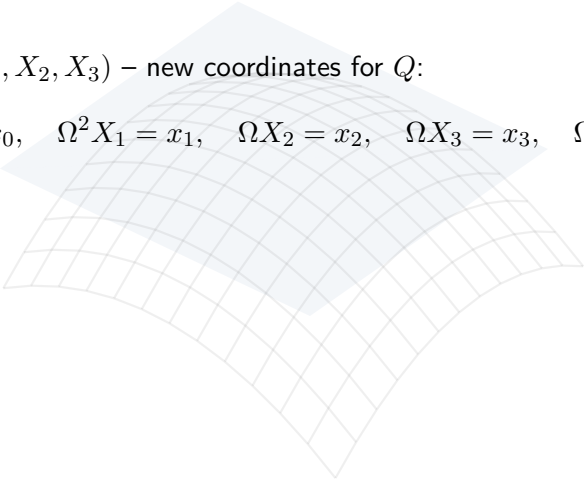
$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & a & b_2 & b_3 \\ 0 & b_2 & -c_{22} & -c_{23} \\ 0 & b_3 & -c_{23} & -c_{33} \end{pmatrix},$$

where a, b_2, \dots are smooth functions of x_0, x_1, x_2, x_3 and

$$\begin{pmatrix} -c_{22} & -c_{23} \\ -c_{23} & -c_{33} \end{pmatrix} - \text{positive definite.}$$

Mathematical procedure

- (X_0, X_1, X_2, X_3) – new coordinates for Q :

$$X_0 = x_0, \quad \Omega^2 X_1 = x_1, \quad \Omega X_2 = x_2, \quad \Omega X_3 = x_3, \quad \Omega > 0.$$


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- define:

$$g_{ij}(\Omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \Omega^2 A & \Omega B_2 & \Omega B_3 \\ 0 & \Omega B_2 & -C_{22} & -C_{23} \\ 0 & \Omega B_3 & -C_{23} & -C_{33} \end{pmatrix},$$

where $A(X_0, X_1, X_2, X_3) = a(x_0, x_1, x_2, x_3), \dots$

Mathematical procedure

- Now:

$$g_{ij}(\Omega)dX_idX_j = \Omega^{-2}g_{ij}dx^idx^j.$$

- $\Omega \rightarrow 0$:

$$2dX_0dX_1 - C_{22}dX_2^2 - 2C_{23}dX_2dX_3 - C_{33}dX_3^2, \quad (1)$$

where C_{22} , C_{23} and C_{33} are functions of X_0 only.

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- The metric form (1) is the familiar Rosen expression for a plane wave – a space-time with a five-parameter group of isometries.

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We established a limiting procedure whereby:

- the immediate neighbourhood of γ' is expanded outwards,
- the metric has been scaled uniformly up at the same time,
- an exact plane wave W_γ is the result.

'Physical' interpretation of the procedure

- We envisage a succession of observers traveling in the space-time M whose world lines approach the null geodesic γ more and more closely; so we picture these observers as traveling with greater and greater speeds, approaching that of light.
- As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all space-time measurements are referred to clock measurements in the standard way), so that in the limit the clocks measure the affine parameter x_0 along γ .
(*Without clock recalibration a degenerate space-time metric would result.*)

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- As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all space-time measurements are referred to clock measurements in the standard way), so that in the limit the clocks measure the affine parameter x_0 along γ .
- In the limit, the observers measure the space-time to have the plane-wave structure W_γ .

Remarks

- For any space-time satisfying the weak energy condition, every null geodesic γ gives rise to a plane wave W_γ admitting an electromagnetic field tensor which, together with the metric of W_γ constitutes a (null) solution of the Einstein-Maxwell equations.
- Why? Because the metric of W_γ is governed by three free real functions of the affine parameter x_0 on γ (whose information is contained in C_{22}, C_{23} and C_{33}):

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 - two of these functions determine the Weyl tensor of W_γ and arise from the complex gravitational null datum Ψ_0 on γ ;
 - the third function determines the Ricci tensor of W_γ and arises from the Ricci component Φ_{00} on γ .

Remarks

- The weak energy condition gives $\Phi_{00} = |\varphi_0|^2 \geq 0$, where φ_0 is a complex function of x_0 .
- The quantities Ψ_0 and φ_0 define the strengths of the null gravitational and null electromagnetic field, respectively, for the plane wave W_γ .

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- They can be chosen as arbitrary functions of x_0 to give a solution of the Einstein-Maxwell equations.
- W_γ is a solution of the vacuum equations for every γ if and only if M is an Einstein space (possibly with non-zero cosmological constant).

Lie group and Lie algebra

- Lie group G

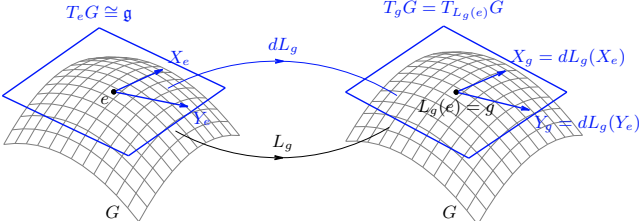


Figure 4: Lie group G and Lie algebra \mathfrak{g}

Lie group and Lie algebra

- Lie group G
- Lie algebra $\mathfrak{g} = Lie G \cong T_e G$

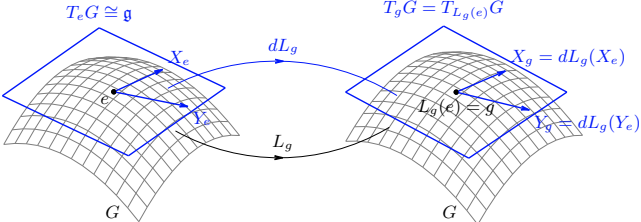


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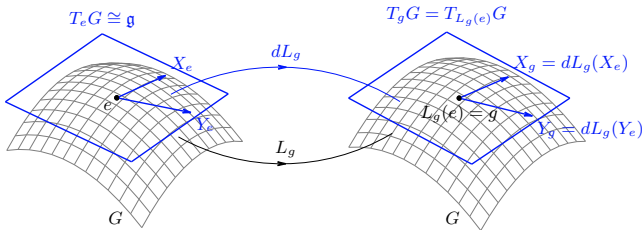


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Parallel and recurrent vector fields

Let G be a Lie group equipped with a left-invariant Lorentzian metric g , and let ∇ denote its Levi-Civita connection.

Parallel vector field

A vector field X on G is **parallel** if

$$\nabla_Y X = 0, \quad \forall Y \in \mathfrak{X}(G).$$

Recurrent vector field

A vector field X on G is **recurrent** if there exists a 1-form ω such that

$$\nabla_Y X = \omega(Y)X, \quad \forall Y \in \mathfrak{X}(G).$$

Waves on Lorentz manifolds

pp-wave (plane-fronted wave with parallel rays)

A Lorentz manifold with parallel light-like vector field X is called **pp-wave** if and only if its curvature tensor satisfies:

$$R(Y_1, Y_2) = 0, \quad \forall Y_1, Y_2 \in X^\perp.$$

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- Type O space-times have vanishing Weyl tensor and hence zero amplitude.

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plane wave

A pp-wave is a **plane wave** if and only if the curvature tensor is parallel in directions orthogonal to the null vector field

$$\nabla_Y R = 0, \quad \forall Y \in X^\perp.$$

Waves on Lorentz manifolds

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pr-wave (plane-fronted wave with recurrent rays)

A Lorentz manifold with recurrent vector field X is called **pr-wave** if and only if its curvature tensor satisfies:

$$R(Y_1, Y_2) = 0, \quad \forall Y_1, Y_2 \in X^\perp.$$

Holonomy of 4-dimensional Lorentzian manifold

Let (M, g) be an Lorentzian manifold and ∇ its Levi-Civita connection. For a point $p \in M$, the **holonomy group** is $Hol_p(M, g) \subset O(1, 3)$ defined as the group of linear transformations of $T_p M$ obtained by parallel transport along loops based at p .

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H -connected holonomy group

- $H = \mathbb{R}^2$ – holonomy of pp-wave;
- $H = \mathbb{R} \ltimes \mathbb{R}^2$ – holonomy of pr-wave.

Example

- $G = H_3 \times \mathbb{R}$, H_3 – Heisenberg group;
- $Lie(G) = \mathfrak{h}_3 \oplus \mathbb{R} : [e_1, e_2] = e_3$;
- $g = dx_1^2 + dx_2^2 - 2dx_4(x_1dx_2 - dx_3)$;
- parallel null vector field $X = \frac{\partial}{\partial x_3}$;
- $R(Y_1, Y_2) = 0$, $Y_1, Y_2 \in X^\perp \rightarrow (G, g)$ – pp-wave;
- $\nabla_X R = 0 \rightarrow (G, g)$ – **plane wave**;
- g is not Einstein metric.

Geodesically equivalent metrics

We aim to study phenomena at vast distances using only telescope observations of many visible objects.



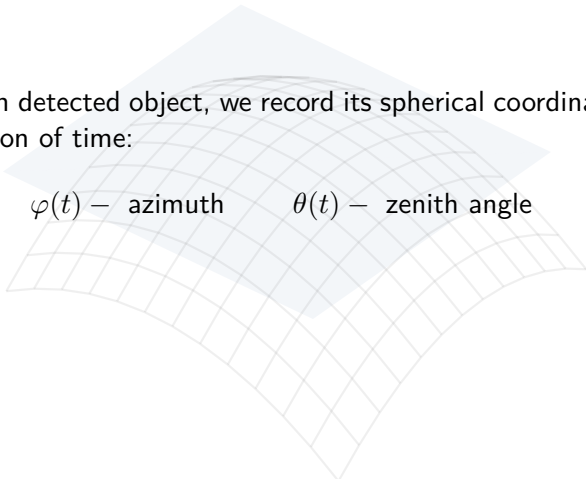
Figure: The Keck observatory, Hawaii

Motivation

- For each detected object, we record its spherical coordinates as a function of time:

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- The observed trajectories are
unparameterized geodesic lines!!!

Definitions

- (M^n, g) – connected (pseudo)-Riemannian manifold, $n \geq 2$
- $g \sim \bar{g}$ – geodesically equivalent = same unparametrized geodesics
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- $\bar{g} = c \cdot g$ – geodesically rigid
 - Weyl (1931) $\Rightarrow c = \text{const}$

Previous results

- Sinjukov (1954): All geodesically equivalent metrics on **symmetric** space are affinely equivalent.
- Petrov (1972): Case of 4-dimensional, Ricci-flat, Lorentz signature: if $g \sim \bar{g}$, then $\nabla = \bar{\nabla}$.

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Affinely equivalent metrics

Construction of affinely equivalent metrics

$v = v^k e_k$ – parallel vector field;

v^* – dual form of v with respect to metric g ;

$\bar{g} \sim g : \bar{g} = \lambda g + \mu v^* \otimes v^*, \lambda, \mu \in \mathbb{R}$.

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 \implies manifold splits (locally)

$|v| = 0 \implies v \subset v^\perp$
 \implies manifold does not have to split

Example: Lorentz metric on $H_3 \times \mathbb{R}$

- $g = dx_1^2 + dx_2^2 - 2dx_4(x_1dx_2 - dx_3)$;
- parallel null vector field $X = \frac{\partial}{\partial x_3}$;
- dual form with respect to metric $X^* = dx_4$;
- $\bar{g} \sim g$, $\lambda, \mu \in \mathbb{R}$:

$$\bar{g} = \lambda(dx_1^2 + dx_2^2 - 2dx_4(x_1dx_2 - dx_3)) + \mu dx_4^2.$$



THANK YOU FOR THE ATTENTION!!!

Questions?