Constraining Extended Gravity by Gravity Probe B and LARES

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based on

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Summary

- ○ ***Extended Gravity***

  The general case: Scalar-tensor-higher-order gravity
  The case of non-commutative spectral geometry

- ○ ***The weak field limit***

  The Newtonian limit
  The post-Newtonian limit

- ○ ***The body motion in the weak gravitational field***

  Circular rotation curves in a spherically symmetric field
  Rotating sources and orbital parameters

- ○ ***Experimental constrains***

- ○ ***Conclusions***

Extended Gravity

Illumination Sq. I

Why extending General Relativity?

- ✓ Several issues in modern Astrophysics ask for new paradigms.
- ✓ No final evidence for Dark Energy and Dark Matter at fundamental level (LHC, astroparticle physics, ground based experiments, LUX...).
- ✓ Such problems could be framed extending GR at infrared scales.
- ✓ GR does not work at ultraviolet scales (no Quantum Gravity).
- ✓ ETGs as minimal extension of GR considering Quantum Fields in Curved Spaces
- ✓ Big issue: Is it possible to find out probes and test-beds for ETGs?
- ✓ Further modes of gravitational waves!
- ✓ Constraints at Newtonian and post-Newtonian level could come from:
  - Geodesic motions around compact objects e.g- SgrA*
  - Lense-Thirring effect
  - Exact torsion-balance experiments
  - Micrograviity experiments from atomic physics
  - Violation of Equivalence Principle (effective masses related to further gravitational degrees of freedom)

Main role of GPB and LARES satellites

The general case: Scalar-tensor-higher-order gravity

ETG means to add further invariants coming from fundamental theories. Consider the action:

$$\mathcal{S} = \int d^4x\sqrt{-g}[f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m],$$

In the metric approach, the gravitational field is described by the metric tensor. The field equations are obtained by varying the action with respect to $g_{\mu\nu}$,

$$\begin{aligned} f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2} g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R \\ + 2f_Y R_\mu{}^\alpha R_{\alpha\nu} - 2[f_Y R^\alpha{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] \\ + [f_Y R_{\alpha\beta}]^{;\alpha\beta} g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} = \mathcal{X}T_{\mu\nu}, \end{aligned}$$

The trace of the field equation

$$\begin{aligned} f_R R + 2f_Y R_{\alpha\beta}R^{\alpha\beta} - 2f + \Box[3f_R + f_Y R] + 2[f_Y R^{\alpha\beta}]_{;\alpha\beta} \\ - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} = \mathcal{X}T, \end{aligned}$$

By varying the action with respect to the scalar field φ, we obtain the Klein-Gordon equation

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_\phi = 0,$$

An example: Non-Commutative Spectral Geometry

For almost-commutative manifolds, the geometry is described by the tensor product M× F of a 4D compact Riemannian manifold M and a discrete non-commutative space F, with M describing the geometry of spacetime and F the internal space of the particle physics model.

The non-commutative nature of F is encoded in the spectral triple (A_F ,H_F ,D_F)

The algebra A_F =C∞(M) of smooth functions on M, playing the role of the algebra of coordinates, is an involution of operators on the finite-dimensional Hilbert space H_F of Euclidean fermions.

The operator D_F is the Dirac operator

$$\partial\!\!\!/_{\mathcal{M}} = \sqrt{-1}\gamma^\mu \nabla^s_\mu$$

on the spin manifold M; it corresponds to the inverse of the Euclidean propagator of fermions and is given by the Yukawa coupling matrix and the Kobayashi-Maskawa mixing parameters.

The algebra A_F has to be chosen so that it can lead to the Standard Model of particle physics, while it must also fulfill non-commutative geometry requirements.

The case of Non-Commutative Spectral Geometry

It is chosen to be

$$\mathcal{A}_{\mathcal{F}} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}).$$

with k=2a; H is the algebra of quaternions, which encodes the non-commutativity of the manifold.

The first possible value for k is 2, corresponding to the Hilbert space of four fermions; it is ruled out from the existence of quarks.

The minimum possible value for k is 4 leading to the correct number of k^2 =16 fermions in each of the three generations.

Higher values of k can lead to particle physics models beyond the Standard Model

The spectral geometry in the product M× F is given by the product rules:

where $L^2(M, S)$ is the Hilbert space of L^2 spinors and D_M is the Dirac operator of the Levi-Cività spin connection on M

$$\mathcal{A} = C^\infty(\mathcal{M}) \oplus \mathcal{A}_F,$$
$$\mathcal{H} = L^2(\mathcal{M}, S) \oplus \mathcal{H}_F,$$
$$\mathcal{D} = \mathcal{D}_M \oplus 1 + \gamma_5 \oplus \mathcal{D}_F.$$

Applying the spectral action principle to the product geometry M×F leads to the NCSG action

$$\mathrm{Tr}(f(D_A/\Lambda)) + (1/2)\langle J\psi, D\psi\rangle$$

split into the bare bosonic action and the fermionic one. Note that D_A =D+A +ϵ'JAJ^{-1} are unimodular inner fluctuations, f is a cutoff function, Λ fixes the energy scale, J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space H of the quarks and leptons.

The case of Non-Commutative Spectral Geometry

In what follows we concentrate on the bosonic part of the action, seen as the bare action at the mass scale Λ which includes the eigenvalues of the Dirac operator that are smaller than the cutoff scale Λ, considered as the grand unification scale.

Using heat kernel methods, the trace $Tr(fD_A/\Lambda)$ can be written in terms of the geometrical Seeley–de Witt coefficients known for any second-order elliptic differential operator, as $\Sigma^{\infty}_{n=0} F_{4-n} \Lambda^{4-n} a_n$ where the function F is defined such that $F(D^2_A) = f(D_A)$.

Considering the Riemannian geometry to be four dimensional, the asymptotic expansion of the trace reads

$$\mathrm{Tr}(f(\mathcal{D}_A/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots + \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots ,$$

where f_k are the momenta of the smooth even test (cutoff) function which decays fast at infinity, and only enters in the multiplicative factors:

$$f_0 = f(0),$$

$$f_2 = \int_0^\infty f(u) u \mathrm{d}u,$$

$$f_4 = \int_0^\infty f(u) u^3 \mathrm{d}u,$$

$$f_{-2k} = (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0)$$

The case of Non-Commutative Spectral Geometry

Since the Taylor expansion of the f function vanishes at zero, the asymptotic expansion of the spectral action reduces to

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{A}}/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

Hence, the cutoff function f plays a role only through its momenta. f_0 , f_2 , f_4 are three real parameters, related to the coupling constants at unification, the gravitational constant, and the cosmological constant, respectively

The NCSG model lives by construction at the grand unification scale, hence providing a framework to study early Universe cosmology

The gravitational part of the asymptotic expression for the bosonic sector of the NCSG action, including the coupling between the Higgs field φ and the Ricci curvature scalar R, in Lorentzian signature, obtained through a Wick rotation in imaginary time, reads

$$\mathcal{S}^{\mathrm{L}}_{\mathrm{grav}} = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{2\kappa_0^2} + \alpha_0 C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + \tau_0 R^{\star} R^{\star} - \xi_0 R |\mathbf{H}|^2 \right];$$

$$\mathbf{H} = (\sqrt{a f_0}/\pi)\phi$$

with a a parameter related to fermion and lepton masses and lepton mixing

At unification scale (set up by Λ), $\alpha_0 = -3f_0/(10\pi^2)$, $\xi_0 = 1/12$.

The case of non-commutative spectral geometry

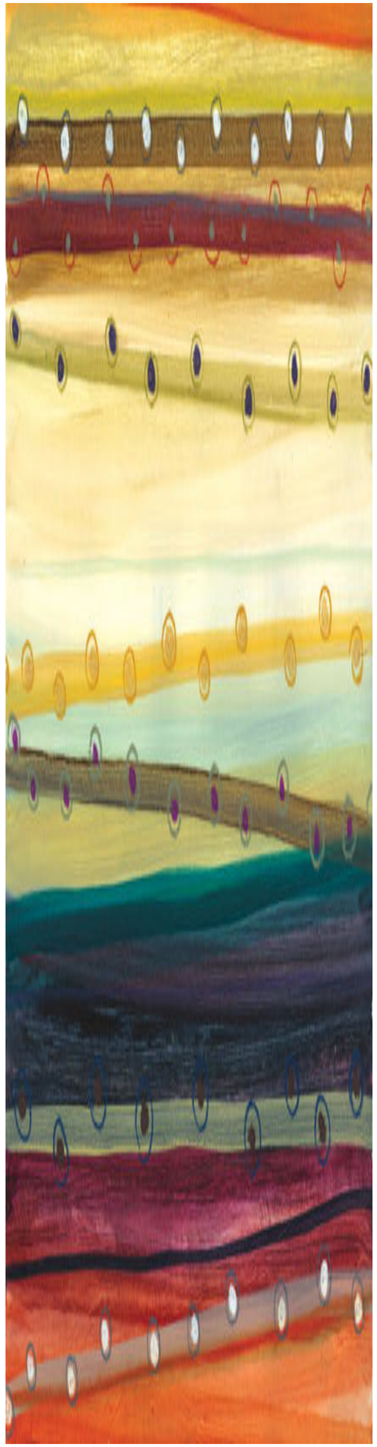
The square of the Weyl tensor can be expressed in terms of R^2 and $R_{\alpha\beta}R^{\alpha\beta}$ as

$$\boxed{C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = 2R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}R^2}$$

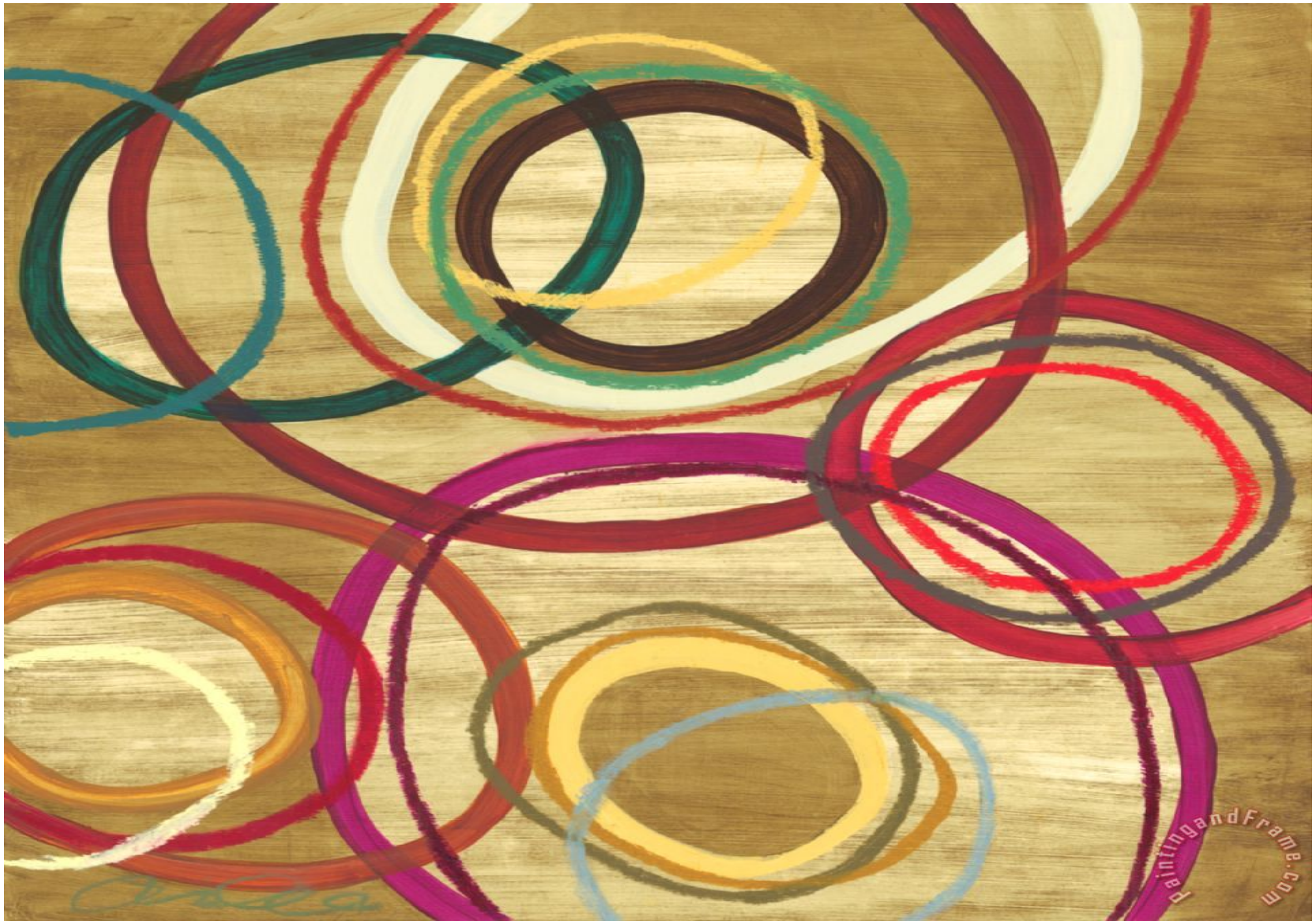
The above action is clearly a particular case of the above action describing a general model of ETG

As we will show, it may lead to effects observable at local scales (in particular at Solar System scales); hence it may be tested against current gravitational data by GPB and LARES.

IN OTHER WORDS, WE CAN USE GPB AND LARES TO TEST FUNDAMENTAL PHYSICS!!!



The weak field limit



Jeni Lee: 21 Tuesday II Bright Circle, 2014

The weak field limit

- *In order to perform the weak-field limit, we have to perturb the field equations in a Minkowski background* $\eta_{\mu\nu}$

- *In a system of gravitationally interacting particles of mass M, the kinetic energy 1/2M* v^2 *is, roughly, of the same order of magnitude as the typical potential energy* $U=GM^2/r$, *with M, r, and v the typical average values of masses, separations, and velocities, respectively, of these particles*

- *As a consequence one has* $v^2 \cong GM/r$ *(for instance, a test particle in a circular orbit of radius r about a central mass M has velocity v given, in Newtonian mechanics, by the exact formula* $v^2=GM/r$*).*

- *The post-Newtonian approximation is a method for obtaining the motion of the system to a higher-than-the- first-order approximation (which coincides with the Newtonian mechanics) with respect to the quantities GM/r, or* v^2*, assumed to be small with respect to the squared speed of light*

- *This approximation is an expansion in inverse powers of the speed of light*

The weak field limit

- *The typical values of the Newtonian gravitational potential Φ are larger (in modulus) than 10^{-5} in the Solar System (in geometrized units, Φ is dimensionless).*
- *Planetary velocities satisfy the condition $v^2 \lesssim -\Phi$, while the matter pressure P experienced inside the Sun and the planets is generally smaller than the matter gravitational energy density $-\rho\,\Phi$; in other words $P/\rho \lesssim -\Phi$*
- *As matter of fact, one can consider that these quantities, as a function of the velocity, give second-order contributions as $-\Phi \sim v^2 \sim O(2)$*
- *Then we can set, as a perturbation scheme of the metric tensor, the following expression*

$$g_{\mu\nu} \sim \begin{pmatrix} 1+g_{tt}^{(2)}(t,\mathbf{x})+g_{tt}^{(4)}(t,\mathbf{x})+\ldots & g_{ti}^{(3)}(t,\mathbf{x})+\ldots \\ g_{ti}^{(3)}(t,\mathbf{x})+\ldots & -\delta_{ij}+g_{ij}^{(2)}(t,\mathbf{x})+\ldots \end{pmatrix} = \begin{pmatrix} 1+2\Phi+2\Xi & 2A_i \\ 2A_i & -\delta_{ij}+2\Psi\delta_{ij} \end{pmatrix}$$

$$\phi \sim \phi^{(0)}+\phi^{(2)}+\ldots = \phi^{(0)}+\varphi,$$

- *Φ, Ψ, ϕ are proportional to the power c^{-2} (Newtonian limit) while A_i is proportional to c^{-3} and Ξ to c^{-4} (post-Newtonian limit)*

The weak field limit

The function f, up to the c^{-4} order, can be developed as

$$f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) = f_R(0,0,\phi^{(0)})R + \frac{f_{RR}(0,0,\phi^{(0)})}{2}R^2 + \frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}(\phi - \phi^{(0)})^2$$
$$+ f_{R\phi}(0,0,\phi^{(0)})R\phi + f_Y(0,0,\phi^{(0)})R_{\alpha\beta}R^{\alpha\beta},$$

while all other possible contributions in f are negligible

The field equations hence read

$$f_R(0,0,\phi^{(0)})\left[R_{tt} - \frac{R}{2}\right] - f_Y(0,0,\phi^{(0)})\triangle R_{tt} - \left[f_{RR}(0,0,\phi^{(0)}) + \frac{f_Y(0,0,\phi^{(0)})}{2}\right]\triangle R - f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = \mathcal{X}T_{tt},$$
$$f_R(0,0,\phi^{(0)})\left[R_{ij} + \frac{R}{2}\delta_{ij}\right] - f_Y(0,0,\phi^{(0)})\triangle R_{ij} + \left[f_{RR}(0,0,\phi^{(0)}) + \frac{f_Y(0,0,\phi^{(0)})}{2}\right]\delta_{ij}\triangle R - f_{RR}(0,0,\phi^{(0)})R_{,ij}$$
$$- 2f_Y(0,0,\phi^{(0)})R^{\alpha}_{(i,j)\alpha} - f_{R\phi}(0,0,\phi^{(0)})(\partial^2_{ij} - \delta_{ij}\triangle)\varphi = \mathcal{X}T_{ij},$$
$$f_R(0,0,\phi^{(0)})R_{ti} - f_Y(0,0,\phi^{(0)})\triangle R_{ti} - f_{RR}(0,0,\phi^{(0)})R_{,ti} - 2f_Y(0,0,\phi^{(0)})R^{\alpha}{}_{(t,i)\alpha} - f_{R\phi}(0,0,\phi^{(0)})\varphi_{,ti}$$
$$= \mathcal{X}T_{ti}, f_R(0,0,\phi^{(0)})R + [3f_{RR}(0,0,\phi^{(0)}) + 2f_Y(0,0,\phi^{(0)})]\triangle R + 3f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = -\mathcal{X}T,$$
$$2\omega(\phi^{(0)})\triangle\varphi + f_{\phi\phi}(0,0,\phi^{(0)})\varphi + f_{R\phi}(0,0,\phi^{(0)})R = 0,$$

where $\triangle$ is the Laplace operator in the flat space

The weak field limit

The geometric quantities $R_{\mu\nu}$ and R are evaluated at the first order with respect to the metric potentials Φ, Ψ and A_i. By introducing the ***effective (masses***

$$m_R{}^2 \doteq -\frac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}, \qquad m_Y{}^2 \doteq \frac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}, \qquad m_\phi{}^2 \doteq -\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})},$$

and setting $f_R(0, 0, \varphi^{(0)})=1$, $\omega(\varphi^{(0)})= 1/2$ for simplicity, we get the complete set of differential equations

$$(\triangle - m_Y{}^2)R_{tt} + \left[\frac{m_Y{}^2}{2} - \frac{m_R{}^2+2m_Y{}^2}{6m_R{}^2}\triangle\right]R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi$$

$$= -m_Y{}^2\mathcal{X}T_{tt}, (\triangle - m_Y{}^2)R_{ij} + \left[\frac{m_R{}^2-m_Y{}^2}{3m_R{}^2}\partial^2_{ij} - \delta_{ij}\left(\frac{m_Y{}^2}{2} - \frac{m_R{}^2+2m_Y{}^2}{6m_R{}^2}\triangle\right)\right]R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})(\partial^2_{ij} - \delta_{ij}\triangle)\varphi$$

$$= -m_Y{}^2\mathcal{X}T_{ij}, (\triangle - m_Y{}^2)R_{ti} + \frac{m_R{}^2-m_Y{}^2}{3m_R{}^2}R_{,ti} + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\varphi_{,ti}$$

$$= -m_Y{}^2\mathcal{X}T_{ti}, (\triangle - m_R{}^2)R - 3m_R{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = m_R{}^2\mathcal{X}T, (\triangle - m_\phi{}^2)\varphi + f_{R\phi}(0,0,\phi^{(0)})R = 0.$$

The components of the Ricci tensor in the weak-field limit read

$$R_{tt} = \frac{1}{2}\triangle g^{(2)}_{tt} = \triangle\Phi,$$

$$R_{ij} = \frac{1}{2}g^{(2)}_{ij,mm} - \frac{1}{2}g^{(2)}_{im,mj} - \frac{1}{2}g^{(2)}_{jm,mi} - \frac{1}{2}g^{(2)}_{tt,ij} + \frac{1}{2}g^{(2)}_{mm,ij} = \triangle\Psi\delta_{ij} + (\Psi - \Phi)_{,ij},$$

$$R_{ti} = \frac{1}{2}g^{(3)}_{ti,mm} - \frac{1}{2}g^{(2)}_{im,mt} - \frac{1}{2}g^{(3)}_{mt,mi} + \frac{1}{2}g^{(2)}_{mm,ti} = \triangle A_i + \Psi_{,ti}.$$

The weak field limit

The energy momentum tensor $T_{\mu\nu}$ can be also expanded

For a perfect fluid, when the pressure is negligible with respect to the mass density ρ, it reads $T_{\mu\nu} = \rho u_\mu u_\nu$ with $u_\sigma u^\sigma = 1$

However, the development starts from the zeroth order; hence $T_{tt} = T^{(0)}{}_{tt} = \rho$, $T_{ij} = T^{(0)}{}_{ij} = 0$ and $T_{ti} = T^{(1)}{}_{ti} = \rho v_i$, where ρ is the density mass and vi is the velocity of the source

Thus, $T_{\mu\nu}$ is independent of metric potentials and satisfies the Bianchi conservation condition $T^{\mu\nu}{}_{,\mu} = 0$

Equations thus read

$$(\triangle - m_Y{}^2)\triangle\Phi + \left[\frac{m_Y{}^2}{2} - \frac{m_R{}^2 + 2m_Y{}^2}{6m_R{}^2}\triangle\right]R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = -m_Y{}^2\mathcal{X}\rho,$$

$$\left\{(\triangle - m_Y{}^2)\triangle\Psi - \left[\frac{m_Y{}^2}{2} - \frac{m_R{}^2 + 2m_Y{}^2}{6m_R{}^2}\triangle\right]R - m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi\right\}\delta_{ij}$$

$$+\left\{(\triangle - m_Y{}^2)(\Psi - \Phi) + \frac{m_R{}^2 - m_Y{}^2}{3m_R{}^2}R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\varphi\right\}_{,ij} = 0,$$

$$\left\{(\triangle - m_Y{}^2)\triangle A_i + m_Y{}^2\mathcal{X}\rho v_i\right\} + \left\{(\triangle - m_Y{}^2)\Psi + \frac{m_R{}^2 - m_Y{}^2}{3m_R{}^2}R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\varphi\right\}_{,ti} = 0,$$

$$(\triangle - m_R{}^2)R - 3m_R{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = m_R{}^2\mathcal{X}\rho,$$

$$(\triangle - m_\phi{}^2)\varphi + f_{R\phi}(0,0,\phi^{(0)})R = 0.$$

The Newtonian limit: Solutions of the fields Φ, ϕ and R

The above equations are a coupled system and, for a pointlike source $\rho(x) = M\,\delta(x)$, admit the solutions

$$\varphi(\mathbf{x}) = \sqrt{\frac{\xi}{3}}\frac{r_g}{|\mathbf{x}|}\frac{e^{-m_R\tilde{k}_R|\mathbf{x}|} - e^{-m_R\tilde{k}_\phi|\mathbf{x}|}}{\tilde{k}_R^2 - \tilde{k}_\phi^2},$$

$$R(\mathbf{x}) = -{m_R}^2\frac{r_g}{|\mathbf{x}|}\frac{(\tilde{k}_R^2 - \eta^2)e^{-m_R\tilde{k}_R|\mathbf{x}|} - (\tilde{k}_\phi^2 - \eta^2)e^{-m_R\tilde{k}_\phi|\mathbf{x}|}}{\tilde{k}_R^2 - \tilde{k}_\phi^2}$$

where r_g is the Schwarzschild radius

$$\tilde{k}_{R,\phi}^2 = \frac{1-\xi+\eta^2 \pm \sqrt{(1-\xi+\eta^2)^2 - 4\eta^2}}{2}$$ ***and*** $$\xi = 3 f_{R\phi}(0,0,\phi^{(0)})^2$$ ***and*** $$\eta = \frac{m_\phi}{m_R}$$

Moreover ξ and η satisfy the condition $$(\eta - 1)^2 - \xi > 0$$

The formal solution of the gravitational potential Φ, reads

$$\Phi(\mathbf{x}) = \frac{-1}{16\pi^2}\int\frac{d^3\mathbf{x}'\,d^3\mathbf{x}''}{|\mathbf{x}-\mathbf{x}'|}\frac{e^{-m_Y|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}'-\mathbf{x}''|}\left[\frac{4{m_Y}^2 - {m_R}^2}{6}\mathcal{X}\rho(\mathbf{x}'') + \frac{{m_Y}^2 - {m_R}^2(1-\xi)}{6}R(\mathbf{x}'') - \frac{{m_R}^4\eta^2}{2\sqrt{3}}\xi^{1/2}\varphi(\mathbf{x}'')\right],$$

The Newtonian limit: Solutions of the fields Φ, ϕ and R

for a pointlike source, it is

$$\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 + g(\xi,\eta)e^{-m_R\tilde{k}_R|\mathbf{x}|} + \left[\frac{1}{3} - g(\xi,\eta)\right]e^{-m_R\tilde{k}_\phi|\mathbf{x}|} - \frac{4}{3}e^{-m_Y|\mathbf{x}|}\right]$$

where

$$g(\xi,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$$

Note that for $f_Y \to 0$ i.e. $m_Y \to \infty$, we obtain the same outcome for the gravitational potential for an $f(R, \varphi)$-theory

The absence of the coupling term between the curvature invariant Y and the scalar field φ, as well as the linearity of the field equations, guarantees that the solution is a linear combination of solutions obtained within an $f(R, \varphi.)$-theory and an $R + Y/mY^2$-theory

The post-Newtonian limit: Solutions of the fields Ψ and A_i

The post-Newtonian limit: Solutions of the fields Ψ and A_i

$$\Psi(\mathbf{x}) = \Phi(\mathbf{x}) + \frac{{m_R}^2 - {m_Y}^2}{12\pi {m_R}^2}\int d^3\mathbf{x}'\,\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}R(\mathbf{x}')$$
$$+\frac{{m_Y}^2\xi^{1/2}}{4\sqrt{3}\pi}\int d^3\mathbf{x}'\,\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\varphi(\mathbf{x}'),$$

which for a pointlike source reads

$$\Psi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\Big[1 - g(\xi,\eta)e^{-m_R\tilde{k}_R|\mathbf{x}|}$$
$$-[1/3 - g(\xi,\eta)]e^{-m_R\tilde{k}_\phi|\mathbf{x}|} - \frac{2}{3}e^{-m_Y|\mathbf{x}|}\Big]$$

obtained by setting $\{\ldots\}_{,ij} = 0$, while one also has $\{\ldots\}\,\delta_{ij} = 0$ leading to

$$\Psi(\mathbf{x}) = -\frac{1}{16\pi^2}\int d^3\mathbf{x}'d^3\mathbf{x}''\,\frac{e^{-m_Y|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}$$
$$\times\Big[\frac{{m_R}^2+2{m_Y}^2}{6}\mathcal{X}\rho(\mathbf{x}'') - \frac{{m_Y}^2-{m_R}^2(1-\xi)}{6}R(\mathbf{x}'')$$
$$+\frac{{m_R}^4\eta^2}{2\sqrt{3}}\xi^{1/2}\varphi(\qquad (1$$

which is however equivalent to solution

The solutions generalize the outcomes of the theory $f(R, R_{\alpha\beta}R^{\alpha\beta})$

The post-Newtonian limit: Solutions of the fields Ψ and A_i

We immediately obtain the solution for A_i, namely

$$A_i(\mathbf{x}) = -\frac{{m_Y}^2\mathcal{X}}{16\pi^2}\int d^3\mathbf{x}'d^3\mathbf{x}''\frac{e^{-m_Y|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}\rho(\mathbf{x}'')v_i''$$

In Fourier space, solution presents the massless pole of general relativity, and the massive one is induced by the presence of the $R^{\alpha\beta}R_{\alpha\beta}$ term

The above solution can be rewritten as the sum of general relativity contributions and massive modes

Since we do not consider contributions inside rotating bodies, we obtain

$$A_i(\mathbf{x}) = -\frac{\mathcal{X}}{4\pi}\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')v_i'}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathcal{X}}{4\pi}\int d^3\mathbf{x}'\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')v_i'$$

For a spherically symmetric system ($|x| = r$) at rest and rotating with angular frequency $\Omega(r)$, the energy momentum tensor T_{ti} is

$$\begin{aligned} T_{ti} &= \rho(\mathbf{x})v_i = T_{tt}(r)[\mathbf{\Omega}(r)\times\mathbf{x}]_i \\ &= \frac{3M}{4\pi\mathcal{R}^3}\Theta(\mathcal{R}-r)[\mathbf{\Omega}(r)\times\mathbf{x}]_i, \end{aligned}$$

where R is the radius of the body and Θ is the Heaviside function

Since only in general relativity and scalar tensor theories the Gauss theorem is satisfied, here we have to consider the potentials Φ, Ψ generated by the ball source with radius R, while they also depend on the shape of the source

The post-Newtonian limit: Solutions of the fields Ψ and A_i

In fact for any term $\propto \frac{e^{-mr}}{r}$ *there is a geometric factor multiplying the Yukawa term, namely*

$$F(m\mathcal{R}) = 3\,\frac{m\mathcal{R}\cosh m\mathcal{R} - \sinh m\mathcal{R}}{m^3\mathcal{R}^3}$$

We thus get

$$\Phi_{\text{ball}}(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 + g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R|\mathbf{x}|} + \left[\frac{1}{3} - g(\xi,\eta)\right]F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi|\mathbf{x}|} - \frac{4F(m_Y\mathcal{R})}{3}e^{-m_Y|\mathbf{x}|}\right]$$

$$\Psi_{\text{ball}}(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 - g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R|\mathbf{x}|} - \left[\frac{1}{3} - g(\xi,\eta)\right]F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi|\mathbf{x}|} - \frac{2F(m_Y\mathcal{R})}{3}e^{-m_Y|\mathbf{x}|}\right].$$

For $\Omega(r)=\Omega_0$, the metric potential reads

$$\mathbf{A}(\mathbf{x}) = -\frac{3MG}{2\pi\mathcal{R}^3}\Omega_0 \times \int d^3\mathbf{x}'\,\frac{1-e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\Theta(\mathcal{R}-r')\mathbf{x}'.$$

Making the approximation $\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \sim \frac{e^{-m_Y r}}{r} + \frac{e^{-m_Y r}(1+m_Y r)\cos\alpha}{r}\frac{r'}{r} + \mathcal{O}\left(\frac{r'^2}{r^2}\right)$

where α is the angle between the vectors x, x', with x = r x where ^x =(sin θ cos φ;, sin θ sin φ, cos θ) and considering only the first order of r'/r, we can evaluate the integration in the vacuum (r > R) as

$$\int d^3\mathbf{x}'\,\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\Theta(\mathcal{R}-r')\mathbf{x}' = \frac{4\pi}{15}\frac{(1+m_Y r)e^{-m_Y r}\mathcal{R}^5}{r^3}\mathbf{x}.$$

The post-Newtonian limit: Solutions of the fields Ψ and A_i

Thus, the field A outside the sphere is

$$\mathbf{A}(\mathbf{x}) = \frac{G}{|\mathbf{x}|^2}\left[1 - (1 + m_Y|\mathbf{x}|)e^{-m_Y|\mathbf{x}|}\right]\hat{\mathbf{x}} \times \mathbf{J},$$

where $J = 2MR^2\Omega_0/5$ is the angular momentum of the ball

The modification with respect to GR has the same feature as the one generated by the pointlike source

From the definition of m_R and m_Y, we note that the presence of a Ricci scalar function [$f_{RR'}(0) \neq 0$] appears only in m_R

Considering only f(R)-gravity ($m_Y \to \infty$), the above solution is unaffected by the modification in the Hilbert-Einstein action.

The body motion in the weak gravitational field

Jeni Lee: Spring Festival II, Jeni Lee, 2013

The body motion in the weak gravitational field

Let us consider the geodesic equations $$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0$$

Where $ds=\sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}$

In terms of the potentials generated by the ball source with radius R, the components of the metric $g_{\mu\nu}$ *read*

$$g_{tt}=1+2\Phi_{\mathrm{ball}}(\mathbf{x})=1-\frac{2GM}{|\mathbf{x}|}\left[1+g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R|\mathbf{x}|}+[1/3-g(\xi,\eta)]F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi|\mathbf{x}|}-\frac{4F(m_Y\mathcal{R})}{3}e^{-m_Y|\mathbf{x}|}\right]$$

$$g_{ti}=2A_i(\mathbf{x})=\frac{2G}{|\mathbf{x}|^2}[1-(1+m_Y|\mathbf{x}|)e^{-m_Y|\mathbf{x}|}]\hat{\mathbf{x}}\times\mathbf{J},$$

$$g_{ij}=-\delta_{ij}+2\Psi_{\mathrm{ball}}(\mathbf{x})\delta_{ij}=-\delta_{ij}-\frac{2GM}{|\mathbf{x}|}\left[1-g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R|\mathbf{x}|}-[1/3-g(\xi,\eta)]F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi|\mathbf{x}|}\right.$$
$$\left.-\frac{2F(m_Y\mathcal{R})}{3}e^{-m_Y|\mathbf{x}|}\right]\delta_{ij},$$

and the non-vanishing Christoffel symbols read

$$\Gamma^t_{ti}=\Gamma^i_{tt}=\partial_i\Phi_{\mathrm{ball}},\qquad \Gamma^i_{tj}=\frac{\partial_iA_j-\partial_jA_i}{2},\qquad \Gamma^i_{jk}=\delta_{jk}\partial_i\Psi_{\mathrm{ball}}-\delta_{ij}\partial_k\Psi_{\mathrm{ball}}-\delta_{ik}\partial_j\Psi_{\mathrm{ball}}.$$

Let us consider some specific motions

Circular rotation curves in a spherically symmetric field

In the Newtonian limit, neglecting the rotating component of the source, leads to the usual equation of motion of bodies

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Phi_{\mathrm{ball}}(\mathbf{x})$$

The study of motion is very simple considering a particular symmetry for mass distribution ρ; otherwise analytical solutions are not available

However, our aim is to evaluate the corrections to the classical motion in the easiest situation, namely the circular motion, in which case we do not consider radial and vertical motions.

The condition of stationary motion on the circular orbit reads

$$v_{\mathrm{c}}(r) = \sqrt{r\frac{\partial\Phi(r)}{\partial r}},$$

where v_c denotes the velocity

Circular rotation curves in a spherically symmetric field

A further remark is needed.

- ***The structure of solutions is mathematically similar to the one of fourth-order gravity $f(R, R_{\alpha\beta}R^{\alpha\beta})$; however there is a fundamental difference regarding the algebraic signs of the Yukawa corrections.***

- ***More precisely, while the Yukawa correction induced by a generic function of the Ricci scalar leads to an attractive gravitational force, and the one induced by the Ricci tensor squared leads to a repulsive one, here the Yukawa corrections induced by a generic function of Ricci scalar and a non-minimally coupled scalar field both have a positive coefficient***

- ***Hence the scalar field gives rise to a stronger attractive force than in f(R)-gravity, which may imply that $f(R, \varphi)$-gravity is a better choice than***
- ***$f(R, R_{\alpha\beta}R^{\alpha\beta})$.-gravity***

- ***However, there is a problem in the limit $|x| \to \infty$: the interaction is scale dependent (the scalar fields are massive) and, in the vacuum, the corrections turn off.***

- ***Thus, at large distances, we recover only the classical Newtonian contribution.***

Circular rotation curves in a spherically symmetric field

The presence of scalar fields makes the profile smooth, a behavior which is apparent in the study of rotation curves.

Let us consider the phenomenological potential $\Phi_{\mathrm{SP}}(r) = -\frac{GM}{r}\left[1 + \alpha e^{-m_{\mathrm{S}} r}\right]$

With α *and* m_S *free parameters, chosen by Sanders in an attempt to fit galactic rotation curves of spiral galaxies in the absence of dark matter, within the modified Newtonian dynamics (MOND) proposal by Milgrom, was further accompanied by a relativistic partner known as the tensor-vector-scalar (TeVeS) model*

The free parameters selected by Sanders were $\alpha \simeq -0.92$ *and* $1/m_S \simeq 40$ *Kpc*

This potential was recently used also for elliptical galaxies

In both cases, assuming a negative value for α*, an almost constant profile for rotation curve is recovered; however there are two issues:*

- *f(R,φ)-gravity does not lead to that negative value of* α*, and secondly the presence of a Yukawa-like correction with negative coefficient leads to a lower rotation curve and only by resetting G one can fit the experimental data.*
- *Only if we consider a massive, non-minimally coupled scalar-tensor theory we get a potential with negative coefficient.*

Circular rotation curves in a spherically symmetric field

In fact setting the gravitational constant equal to

$$G_0 = \frac{2\omega(\phi^{(0)})\phi^{(0)} - 4}{2\omega(\phi^{(0)})\phi^{(0)} - 3} \frac{G_\infty}{\phi^{(0)}}$$

where G_∞ is the gravitational constant as measured at infinity, and imposing

$$\alpha^{-1} = 3 - 2\omega(\phi^{(0)})\phi^{(0)}$$

the potential becomes

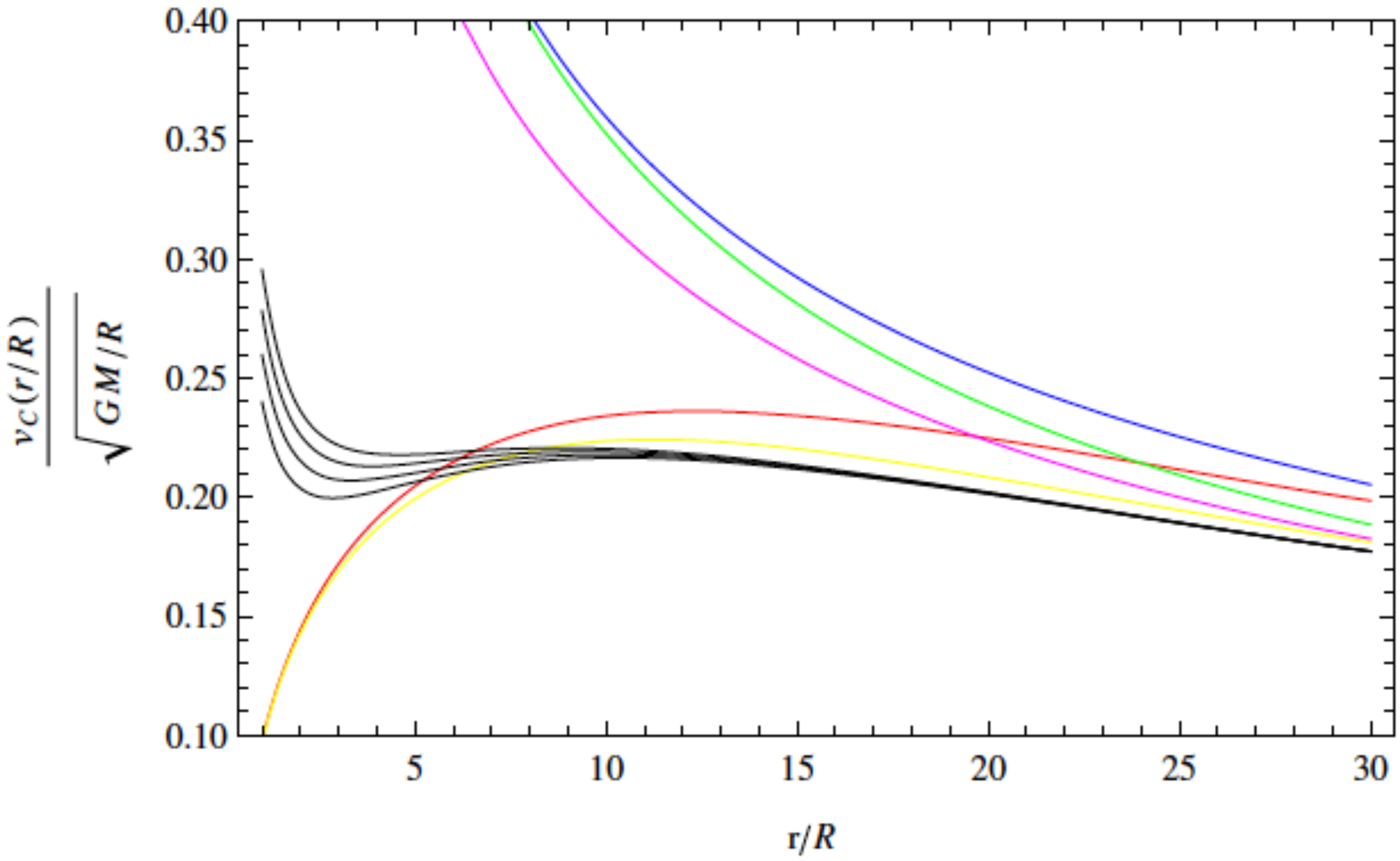
$$\boxed{\Phi(r) = -\frac{G_\infty M}{r} \left\{1 + \alpha e^{-\sqrt{1-3\alpha}\, m_\phi r}\right\}}$$

and then the Sanders potential can be recovered.

In Fig. below we show the radial behavior of the circular velocity induced by the presence of a ball source in the case of the Sanders potential and of potentials shown in next Table.

TABLE I. Table of fourth-order gravity models analyzed in the Newtonian limit for gravitational potentials generated by a pointlike source Eq. (17). The range of validity of cases C, D is $(\eta-1)^2-\xi>0$. We set $f_R(0,0,\phi^{(0)})=1$.

Case	Theory	Gravitational potential	Free parameters
A	$f(R)$	$-\frac{GM}{\lvert\mathbf{x}\rvert}[1+\frac{1}{3}e^{-m_R\lvert\mathbf{x}\rvert}]$	$m_R{}^2=-\frac{1}{3f_{RR}(0)}$
B	$f(R,R_{\alpha\beta}R^{\alpha\beta})$	$-\frac{GM}{\lvert\mathbf{x}\rvert}[1+\frac{1}{3}e^{-m_R\lvert\mathbf{x}\rvert}-\frac{4}{3}e^{-m_Y\lvert\mathbf{x}\rvert}]$	$m_R{}^2=-\frac{1}{3f_{RR}(0,0)+2f_Y(0,0)}$ $m_Y{}^2=\frac{1}{f_Y(0,0)}$
C	$f(R,\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{GM}{\lvert\mathbf{x}\rvert}[1+g(\xi,\eta)e^{-m_R\tilde{k}_R\lvert\mathbf{x}\rvert}$ $+[1/3-g(\xi,\eta)]e^{-m_R\tilde{k}_\phi\lvert\mathbf{x}\rvert}]$	$m_R{}^2=-\frac{1}{3f_{RR}(0,\phi^{(0)})}$ $m_\phi{}^2=-\frac{f_{\phi\phi}(0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $\xi=\frac{3f_{R\phi}(0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta=\frac{m_\phi}{m_R}$ $g(\xi,\eta)=\frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$ $\tilde{k}^2_{R,\phi}=\frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$
D	$f(R,R_{\alpha\beta}R^{\alpha\beta},\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{GM}{\lvert\mathbf{x}\rvert}[1+g(\xi,\eta)e^{-m_R\tilde{k}_R\lvert\mathbf{x}\rvert}$ $+[1/3-g(\xi,\eta)]e^{-m_R\tilde{k}_\phi\lvert\mathbf{x}\rvert}-\frac{4}{3}e^{-m_Y\lvert\mathbf{x}\rvert}]$	$m_R{}^2=-\frac{1}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}$ $m_Y{}^2=\frac{1}{f_Y(0,0,\phi^{(0)})}$ $m_\phi{}^2=-\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $\xi=\frac{3f_{R\phi}(0,0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta=\frac{m_\phi}{m_R}$ $g(\xi,\eta)=\frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$ $\tilde{k}^2_{R,\phi}=\frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$



The circular velocity of a ball source of mass M and radius R, with the potentials of Table I. We indicate case A by a green line, case B by a yellow line, case D by a red line, case C by a blue line, and the GR case by a magenta line. The black lines correspond to the Sanders model for $-0.95 < \alpha < -0.92$. The values of free parameters are $\omega(\varphi^{(0)})$.. . $-1/2$,
$\Xi = -5$, $\eta = .3$, $m_Y = 1.5 * m_R$, $m_S = 1.5 * m_R$, $m_R = .1 * R^{-1}$.

Rotating sources and orbital parameters

Considering the geodesic equations with the Christoffel symbols, we obtain

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{tt} + 2\Gamma^i_{tj}\frac{dx^j}{ds} = 0,$$

which in the coordinate system J = (0, 0, J) reads

$$\ddot{x} + \frac{GM}{r^3}x = -\frac{GM\Lambda(r)}{r^3}x + \frac{2GJ}{r^5}\left\{\zeta(r)\left[\left(x^2+y^2-2z^2\right)\dot{y} + 3yz\dot{z}\right] + 2\Sigma(r)L_x z\right\}$$

$$\ddot{y} + \frac{GM}{r^3}y = -\frac{GM\Lambda(r)}{r^3}y - \frac{2GJ}{r^5}\left\{\zeta(r)\left[\left(x^2+y^2-2z^2\right)\dot{x} + 3xz\dot{z}\right] - 2\Sigma(r)L_y z\right\},$$

$$\ddot{z} + \frac{GM}{r^3}z = -\frac{GM\Lambda(r)}{r^3}z + \frac{6GJ}{r^5}\left\{\zeta(r) + \frac{2}{3}\Sigma(r)\right\}L_z z,$$

where

$$\Lambda(r) \doteq g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})(1+m_R\tilde{k}_R r)e^{-m_R\tilde{k}_R r} + [1/3 - g(\xi,\eta)]F(m_R\tilde{k}_\phi\mathcal{R})(1+m_R\tilde{k}_\phi r)e^{-m_R\tilde{k}_\phi r} - \frac{4F(m_Y\mathcal{R})}{3}(1+m_Y r)e^{-m_Y r},$$

$$\zeta(r) \doteq 1 - [1 + m_Y r + (m_Y r)^2]e^{-m_Y r},$$

$$\Sigma(r) \doteq (m_Y r)^2 e^{-m_Y r},$$

with L_x, L_y and L_z the components of the angular momentum

Rotating sources and orbital parameters

The first terms in the right-hand side of the above equation, depending on the three parameters m_R, m_Y and m_φ, represent the Extended Gravity (EG) modification of the Newtonian acceleration.

The second terms in these equations, depending on the angular momentum J and the EG parameters m_R, m_Y and m_φ, correspond to DRAGGING CONTRIBUTIONS

The case $m_R \to \infty$, $m_Y \to \infty$ and $m_\varphi \to 0$ leads to $\Lambda(r) \to 0$, $\xi(r) \to 1$ and $\Sigma(r) \to 0$, and hence one recovers the familiar results of GR

These additional gravitational terms can be considered as perturbations of Newtonian gravity, and their effects on planetary motions can be calculated within the usual perturbation schemes assuming the Gauss equations

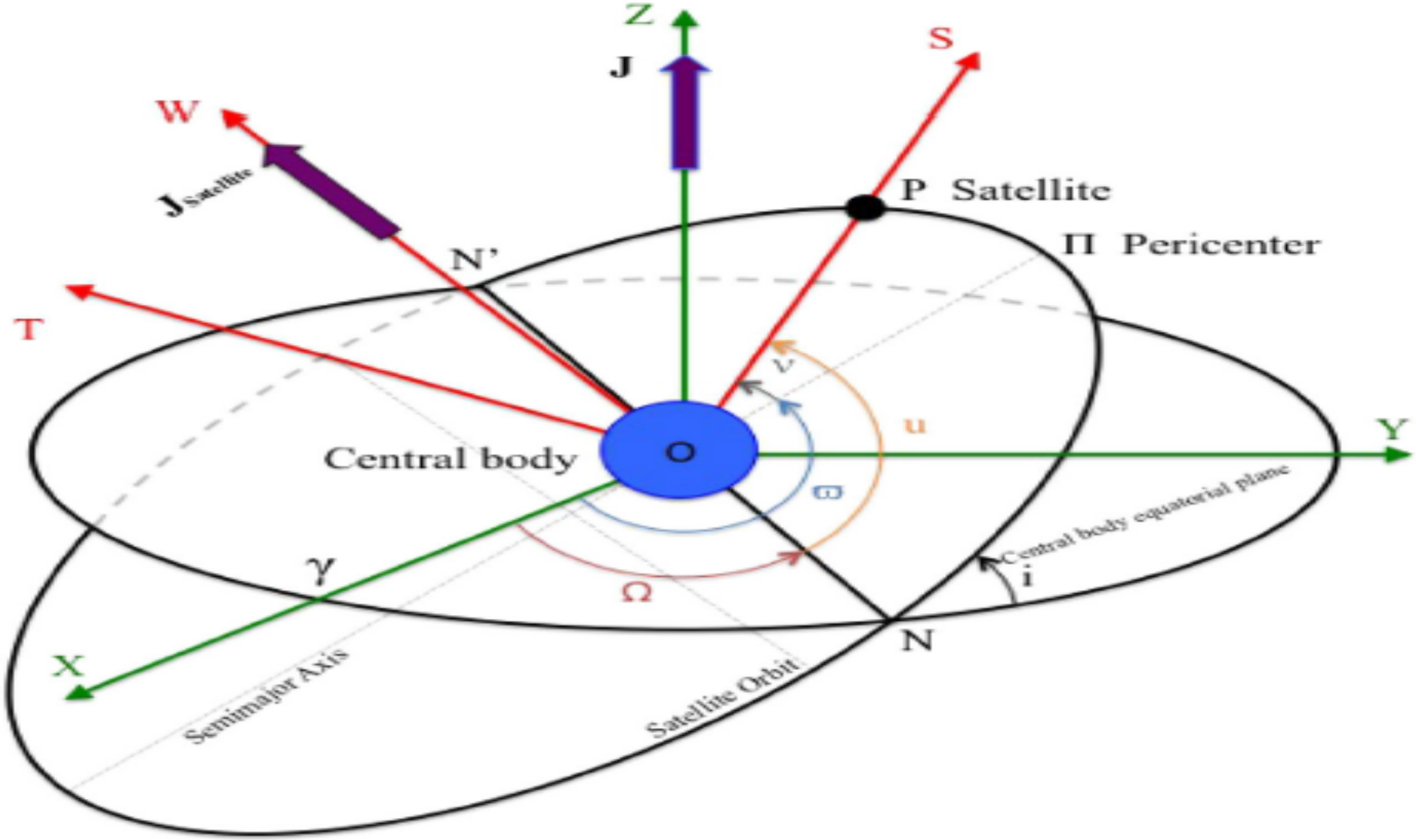
Rotating sources and orbital parameters

Let us consider the right-hand side of the above equations as the components (A_x, A_y, A_z) of the perturbing acceleration in the system (X, Y, Z) (see next Fig.), with X the axis passing through the vernal equinox γ, Y the transversal axis, and Z the orthogonal axis parallel to the angular momentum J of the central body

In the system (S,T,W), the three components can be expressed as (A_s, A_t, A_w), with S the radial axis, T the transversal axis, and W the orthogonal one

We will adopt the standard notation:

- *a is the semimajor axis;*
- *e is the eccentricity*
- *$p=a(1-e^2)$ is the semilatus rectum;*
- *i is the inclination;*
- *Ω is the longitude of the ascending node N;*
- *$\omega\sim$ is the longitude of the pericenter Π;*
- *M^0 is the longitude of the satellite at time $t = 0$;*
- *ν is the true anomaly;*
- *u is the argument of the latitude given by $u = \nu + \omega\sim - \Omega$;*
- *n is the mean daily motion equal to $n=(GM/a^3)^{1/2}$;*
- *and C is twice the velocity, namely* $C = r^2\dot{\nu}a^2(1-e^2)^{1/2}$



$i = \sphericalangle YN\Pi$ is the inclination; $\Omega = \sphericalangle XON$ is the longitude of the ascending node N; $\omega\sim$ = broken$\sphericalangle XO\Pi$ is the longitude of the pericenter Π; $\nu = \sphericalangle \Pi OP$ is the true anomaly; $u = \sphericalangle \Omega OP = \nu + \omega\sim - \Omega$ is the argument of the latitude; J is the angular momentum of rotation of the central body; and $J_{Satellite}$ is the angular momentum of revolution of a satellite around the central body.

Rotating sources and orbital parameters

The transformation rules between the coordinates frames (X, Y, Z) and (S, T,W) are

$$x = r(\cos u \cos\Omega - \sin u \sin\Omega \cos i),$$
$$y = r(\cos u \sin\Omega + \sin u \cos\Omega \cos i),$$
$$z = r \sin u \sin i$$
$$r = \frac{p}{1 + e\cos\nu},$$

and the components of the angular momentum obey the equations

$$L_x = y\dot{z} - z\dot{y} = C \sin i \sin\Omega,$$
$$L_y = z\dot{x} - x\dot{z} = -C \cos\Omega \sin i,$$
$$L_z = x\dot{y} - y\dot{x} = C \cos i.$$

The components of the perturbing acceleration in the (S, T,W) system read

$$A_s = -\frac{GM\Lambda(r)}{r^2} + \frac{2GJC\cos i}{r^4}\zeta(r),$$
$$A_t = -\frac{2GJCe\cos i \sin\nu}{pr^3}\zeta(r),$$
$$A_w = \frac{2GJC\sin i}{r^4}\left[\left(\frac{re\sin\nu\cos u}{p} + 2\sin u\right)\zeta(r) + 2\sin u\Sigma(r)\right].$$

The As component has two contributions: the former one results from the modified Newtonian potential $\Phi_{ball}(x)$, while the latter one results from the gravitomagnetic field A_i and it is a higher order term than the first one

Note that the components A_t and A_w depend only on the gravito-magnetic field

Rotating sources and orbital parameters

The Gauss equations for the variations of the six orbital parameters, resulting from the perturbing acceleration with components A_x , A_y ,A_z , read

$$\frac{da}{dt} = \dot{a}_{\mathrm{EG}} = \frac{2eGM\Lambda(r)\sin\nu}{n\sqrt{1-e^2}C}\dot{\nu},$$

$$\frac{de}{dt} = \dot{e}_{\mathrm{GR}} + \dot{e}_{\mathrm{EG}} = \frac{\sqrt{1-e^2}GM\Lambda(r)\sin\nu}{naC}\dot{\nu} + \dot{e}_{\mathrm{GR}}[1 - e^{-m_Y r}(1 + m_Y r + (m_Y r)^2)],$$

$$\frac{d\Omega}{dt} = \dot{\Omega}_{\mathrm{GR}} + \dot{\Omega}_{\mathrm{EG}} = \dot{\Omega}_{\mathrm{GR}}\{1 - e^{-m_Y r}[1 + m_Y r + (1 + f(\nu, u, e))(m_Y r)^2]\},$$

$$\frac{di}{dt} = \dot{i}_{\mathrm{GR}} + \dot{i}_{\mathrm{EG}} = \dot{i}_{\mathrm{GR}}\{1 - e^{-m_Y r}[1 + m_Y r + (1 + f(\nu, u, e))(m_Y r)^2]\},$$

$$\frac{d\tilde{\omega}}{dt} = \dot{\tilde{\omega}}_{\mathrm{GR}} + \dot{\tilde{\omega}}_{\mathrm{EG}} = -\frac{\sqrt{1-e^2}GM\Lambda(r)\cos\nu}{naeC}\dot{\nu} + \dot{\tilde{\omega}}_{\mathrm{GR}}[1 - e^{-m_Y r}(1 + m_Y r + (m_Y r)^2)] - 2\sin^2\frac{i}{2}\dot{\Omega}_{\mathrm{GR}}f(\nu, u, e)\Sigma(r),$$

$$\frac{dM^0}{dt} = \dot{M^0}_{\mathrm{GR}} + \dot{M^0}_{\mathrm{EG}} = -\frac{GM\Lambda(r)}{naC}\left[\frac{2r}{a} + \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cos\nu\right]\dot{\nu} + \dot{M^0}_{\mathrm{GR}}[1 - e^{-m_Y r}(1 + m_Y r + (m_Y r)^2)]$$
$$- 2\sin^2\frac{i}{2}\dot{\Omega}_{\mathrm{GR}}f(\nu, u, e)\Sigma(r),$$

where

$$\dot{e}_{\mathrm{GR}} = \frac{2GJ\cos i\sin\nu}{aC}\dot{\nu},$$

$$\dot{\Omega}_{\mathrm{GR}} = \frac{2GJ\sin u}{pC}[e\sin\nu\cos u + 2(1 + e\cos\nu)\sin u]\dot{\nu},$$

$$\dot{i}_{\mathrm{GR}} = \frac{2GJ\cos u\sin i}{Cp}[e\sin\nu\cos u + 2(1 + e\cos\nu)\sin u]\dot{\nu},$$

$$\dot{\tilde{\omega}}_{\mathrm{GR}} = -\frac{2GJ\cos i}{aC}\left(2 + \frac{1+e^2}{e}\cos\nu\right)\dot{\nu} + 2\sin^2\frac{i}{2}\dot{\Omega}_{\mathrm{GR}},$$

$$\dot{\mathcal{M}}^0{}_{\mathrm{GR}} = -\frac{4GJ\cos i}{na^2p}(1 + e\cos\nu)\dot{\nu} + \frac{e^2}{1+\sqrt{1-e^2}}\dot{\tilde{\omega}}_{\mathrm{GR}}$$
$$+ 2\sqrt{1-e^2}\sin^2\frac{i}{2}\dot{\Omega}_{\mathrm{GR}},$$

$$f(\nu, u, e) = \frac{1 + e\cos\nu}{1 + e(\frac{\sin\nu\cot u}{2} + \cos\nu)}.$$

we have derived the corresponding equations of the six orbital parameters for extended gravity, with the dynamics of a; e; ω~ ; L^0 depending mainly on the terms related to the modifications of the Newtonian potential, while the dynamics of Ω and i depend only on the dragging terms

Rotating sources and orbital parameters

Considering an almost circular orbit ($e \ll 1$), we integrate the Gauss equations with respect to the only anomaly ν, from 0 to $\nu(t) = nt$, since all other parameters have a slower evolution than ν, hence they can be considered as constraints with respect to ν At first order we get

$$\Delta a(t) = 0,$$
$$\Delta e(t) = 0,$$
$$\Delta i(t) = \frac{GJe^2 \sin i}{na^3} e^{-m_Y p}(m_Y p)^2 \left[1 + \frac{(m_Y p)^2}{2}(m_Y p - 4)\right. \times \sin(\tilde{\omega}(t) - \Omega(t))\nu(t) + \mathcal{O}(e^4),$$
$$\Delta\Omega(t) = \frac{2GJ}{na^3}[1 - e^{-m_Y p}(1 + m_Y p + 2(m_Y p)^2)]\nu(t) + \mathcal{O}(e^2),$$
$$\Delta\tilde{\omega}(t) = \left\{\frac{\tilde{\Lambda}(p)}{2} - \frac{2GJ}{na^3}[3\cos i - 1 + e^{-m_Y p}(1 + m_Y p + \frac{3}{2}(m_Y p)^2 - (3 + 3m_Y p + 3(m_Y p)^2 + \frac{1}{12}(m_Y p)^3)\cos i)]\right\}\nu(t) + \mathcal{O}(e^2),$$
$$\Delta\mathcal{M}^0(t) = \left\{2\Lambda(p) - \frac{2GJ}{na^3}[3\cos i - 1 - e^{-m_Y p}(1 + m_Y p + 2(m_Y p)^2)(\cos i - 1)]\right\}\nu(t) + \mathcal{O}(e^2),$$

where

$$\tilde{\Lambda}(p) \doteq g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})(m_R\tilde{k}_R p)^2 e^{-m_R\tilde{k}_R p} + [1/3 - g(\xi,\eta)]F(m_R\tilde{k}_\phi\mathcal{R})(m_R\tilde{k}_\phi p)^2 e^{-m_R\tilde{k}_\phi p} - \frac{4F(m_Y\mathcal{R})}{3}(m_Y p)^2 e^{-m_Y p}.$$

We hence notice that the contributions to the semimajor axis a and eccentricity e vanish, as in GR, while there are nonzero contributions to i, Ω, ω~ and M^0. In particular, the contributions to the inclination i and the longitude of the ascending node Ω depend only on the drag effects of the rotating central body, while the contributions to the pericenter longitude ω~ and mean longitude at M^0 depend also on the modified Newtonian potential

Rotating sources and orbital parameters

Finally, note that in the ETG models we have considered here, the inclination i has a nonzero contribution, in contrast to the result obtained within GR, and also
$\Delta\,\omega\,(t) \neq \Delta M^0(t)$, given by

$$\Delta\tilde{\omega}(t) - \Delta\mathcal{M}^0(t) \simeq \left\{ \frac{\tilde{\Lambda}(p) - 4\Lambda(p)}{2} + \frac{2GJ}{na^3} e^{-m_Y p} \left[\frac{(m_Y p)^2}{2} + \left(2 + 2m_Y p + (m_Y p)^2 + \frac{(m_Y p)^3}{12} \right) \cos i \right] \right\} \nu(t) + \mathcal{O}(e^2).$$

In the limit $m_R \rightarrow \infty$; $m_Y \rightarrow \infty$ and $m_\varphi \rightarrow 0$, we obtain the well-known results of GR.

Experimental constrains

Jeni Lee: Flight of Fancy V , 2012

Experimental constrains

The orbiting gyroscope precession can be split into a part generated by the metric potentials, Φ and Ψ, and one generated by the vector potential A

The equation of motion for the gyrospin three-vector S is

$$\frac{d\mathbf{S}}{dt} = \left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{G}} + \left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{LT}}$$

where the geodesic and Lense-Thirring precessions are

$$\left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{G}} = \Omega_{\mathrm{G}} \times \mathbf{S} \quad \text{with} \quad \Omega_{\mathrm{G}} = \frac{\nabla(\Phi + 2\Psi)}{2} \times \mathbf{v}$$

$$\left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{LT}} = \Omega_{\mathrm{LT}} \times \mathbf{S} \quad \text{with} \quad \Omega_{\mathrm{LT}} = \frac{\nabla \times \mathbf{A}}{2}$$

The geodesic precession, ΩG can be written as the sum of two terms, one obtained with GR and the other being the extended gravity contribution

Then we have

$$\Omega_{\mathrm{G}} = \Omega_{\mathrm{G}}^{(\mathrm{GR})} + \Omega_{\mathrm{G}}^{(\mathrm{EG})}$$

where

$$\Omega_{\mathrm{G}}^{(\mathrm{GR})} = \frac{3GM}{2|\mathbf{x}|^3}\mathbf{x} \times \mathbf{v},$$

$$\Omega_{\mathrm{G}}^{(\mathrm{EG})} = -\left[g(\xi,\eta)(m_R\tilde{k}_R r + 1)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R r} + \frac{8}{3}(m_Y r + 1)F(m_Y\mathcal{R})e^{-m_Y r}\right.$$
$$\left. + \left[\frac{1}{3} - g(\xi,\eta)\right](m_R\tilde{k}_\phi r + 1)F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi r}\right]\frac{\Omega_{\mathrm{G}}^{(\mathrm{GR})}}{3},$$

Where $|x| = r$

Experimental constrains

Similarly one has $$\Omega_{\mathrm{LT}} = \Omega_{\mathrm{LT}}^{(\mathrm{GR})} + \Omega_{\mathrm{LT}}^{(\mathrm{EG})}$$

with $\Omega_{\mathrm{LT}}^{(\mathrm{GR})} = \dfrac{G}{2r^3}\mathbf{J}$ *and* $\Omega_{\mathrm{LT}}^{(\mathrm{EG})} = -e^{-m_Y r}(1 + m_Y r + m_Y{}^2 r^2)\Omega_{\mathrm{LT}}^{(\mathrm{GR})}$

where we have assumed that, on the average, $<(J\cdot x).x_i>$.

The Gravity Probe B (GPB) satellite contains a set of four gyroscopes and has tested two predictions of GR: the geodetic effect and frame-dragging (Lense-Thirring effect)

The tiny changes in the direction of spin gyroscopes, contained in the satellite orbiting at h = 650 km of altitude and crossing directly over the poles, have been measured with extreme precision

The values of the geodesic precession and the Lense-Thirring precession, measured by the Gravity Probe B satellite and those predicted by GR, are given in

Effect	Measured (mas/y)	Predicted (mas/y)
Geodesic precession	6602 ± 18	6606
Lense-Thirring precession	37.2 ± 7.2	39.2

Experimental constrains

Imposing the constraint $|\Omega^{(EG)}{}_G| \lesssim \delta\Omega_G$ *and* $|\Omega^{(EG)}{}_{LT}| \lesssim \delta\Omega_{LT}$, *with* $r^* = R_\oplus + h$ *where* $R_\oplus$ *is the radius of the Earth and* h *= 650 km is the altitude of the satellite, we get*

$$g(\xi,\eta)(m_R\tilde{k}_R r^* + 1)F(m_R\tilde{k}_R R_\oplus)e^{-m_R\tilde{k}_R r^*} + [1/3 - g(\xi,\eta)](m_R\tilde{k}_\phi r^* + 1)F(m_R\tilde{k}_\phi R_\oplus)e^{-m_R\tilde{k}_\phi r^*}$$
$$+\frac{8}{3}(m_Y r^* + 1)F(m_Y R_\oplus)e^{-m_Y r^*} \lesssim \frac{3\delta|\Omega_{\mathrm{G}}|}{|\Omega_{\mathrm{G}}^{(\mathrm{GR})}|} \simeq 0.008,$$

$$(1 + m_Y r^* + m_Y{}^2 r^{*2})e^{-m_Y r^*} \lesssim \frac{\delta|\Omega_{\mathrm{LT}}|}{|\Omega_{\mathrm{LT}}^{(\mathrm{GR})}|} \simeq 0.19,$$

since, from the experiments, we have $|\Omega^{(GR)}{}_G|$ *= 6606 mas and* $\delta|\Omega_G|$*=18 mas,* $|\Omega^{(GR)}{}_{LT}|$ *= 37.2 mas and* $\delta|\Omega_{LT}|$ *= 7.2 mas*

We thus obtain that $m_Y \geq 7.3 \times 10^{-7} m^{-1}$

Experimental constrains

The Laser Relativity Satellite (LARES) mission of the Italian Space Agency is designed to test the frame dragging and the Lense-Thirring effect, to within 1% of the value predicted in the framework of GR

The body of this satellite has a diameter of about 36.4 cm and weights about 400 kg

It was inserted in an orbit with 1450 km of perigee, an inclination of 69.5 ± 1 degrees and eccentricity 9.54×10^{-4}

It allows us to obtain a ***stronger constraint*** *for m_Y:*

$$\boxed{(1 + m_Y r^* + {m_Y}^2 r^{*2}) e^{-m_Y r^*} \lesssim \frac{\delta|\Omega_{\rm LT}|}{|\Omega_{\rm LT}^{\rm (GR)}|} \simeq 0.01}$$

From which we obtain $m_Y \geq 1.2 \times 10^{-6} m^{-1}$

Experimental constrains

In the specific case of the Non-Commutative Spectral Geometry, the above quantities become for $m_R \to \infty$,

$$m_Y = \sqrt{\frac{5\pi^2(k_0^2\mathbf{H}^{(0)}-6)}{36 f_0 k_0^2}}$$

and $m_\phi = 0$ *implying that* $\xi = \frac{a f_0 (\mathbf{H}^{(0)})^2}{12\pi^2}$

$$\eta = 0, \qquad g(\xi,\eta) = \frac{a f_0(\mathbf{H}^{(0)})^2 + 12\pi^2}{6|a f_0(\mathbf{H}^{(0)})^2 - 12\pi^2|} + \frac{1}{6} \qquad \text{and} \qquad \tilde{k}^2_{R,\phi} = 1 - \frac{a f_0(\mathbf{H}^{(0)})^2}{12\pi^2}$$

The first relation $\frac{8}{3}(m_Y r^* + 1)F(m_Y R_\oplus)e^{-m_Y r^*} \lesssim 0.008;$

hence the constraint on m_Y *imposed from GPB is* $m_Y > 7.1 \times 10^{-5}\ \mathrm{m}^{-1}$

whereas the LARES experiment implies $\boxed{m_Y > 1.2 \times 10^{-6}\ \mathrm{m}^{-1}}$

A bound similar to the one obtained earlier by using binary pulsars, or the Gravity Probe B data.

However, a more stringent constraint has been obtained using torsion balance experiments

using results from laboratory experiments designed to test the fifth force, one arrives to the tightest constraint $m_Y > 10^4\ m^{-1}$

Experimental constrains

In conclusion, using data from the Gravity Probe B and LARES missions, we obtain similar constraints on m_Y, a result that one could have anticipated since both experiments are designed to test the same type of physical phenomena

However, by using the stronger constraint for m_Y, namely $m_Y > 10^4\ m^{-1}$, we observe that the modifications to the orbital parameters induced by Non-Commutative Spectral Geometry are indeed small, confirming the consistency between the predictions of NCSG, as a gravitational theory beyond GR, and Gravity Probe B and LARES measurements

This results show that space-based experiments can be used to test extensively parameters of fundamental theories

Conclusions

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Conclusions

- *In the context of ETGs, we have studied the linearized field equations in the limit of weak gravitational fields and small velocities generated by rotating gravitational sources, aimed to constrain the free parameters, which can be seen as effective masses (or lengths).*

- *We have studied the precession of spin of a gyroscope orbiting around a rotating gravitational source.*

- *Such a gravitational field gives rise, according to GR predictions, to geodesic and Lense-Thirring processions, the latter being strictly related to the off-diagonal terms of the metric tensor generated by the rotation of the source*

- *We have focused in particular on the gravitational field generated by the Earth, and on the recent experimental results obtained by the Gravity Probe B and LARES satellites, which tested the geodesic and Lense-Thirring spin precessions with high precision.*

- *In particular, we have calculated the corrections of the precession induced by scalar, tensor and curvature corrections.*

Conclusions

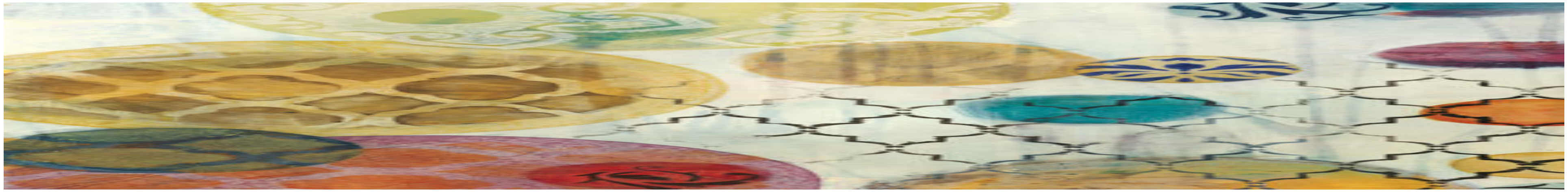
- *Considering an almost circular orbit, we integrated the Gauss equations and obtained the variation of the parameters at first order with respect to the eccentricity.*

- *We have shown that the induced EG effects depend on the effective masses m_R, m_Y and m_φ, while the non validity of the Gauss theorem implies that these effects also depend on the geometric form and size of the rotating source.*

- *Requiring that the corrections be within the experimental errors, we then imposed constraints on the free parameters of the considered EG model. Merging the experimental results of Gravity Probe B and LARES, our results can be summarized as follows:*

$$\boxed{\begin{aligned} & g(\xi,\eta)(m_R\tilde{k}_R r^* + 1)F(m_R\tilde{k}_R R_\oplus)e^{-m_R\tilde{k}_R r^*} \\ & \quad + [1/3 - g(\xi,\eta)](m_R\tilde{k}_\phi r^* + 1)F(m_R\tilde{k}_\phi R_\oplus)e^{-m_R\tilde{k}_\phi r^*} \\ & \quad + \frac{8}{3}(m_Y r^* + 1)F(m_Y R_\oplus)e^{-m_Y r^*} \lesssim 0.008, \end{aligned}}$$

and $m_Y \geq 1.2 \times 10^{-6} m^{-1}$



Conclusions

- *The field equation for the potential A_i, is time independent provided the potential Φ is time independent.*
- *This aspect guarantees that the solution does not depend on the masses m_R and m_φ and, in the case of $f\,(R, \varphi)$. gravity, the solutions the same as in GR*
- *In the case of spherical symmetry, the hypothesis of a radially static source is no longer considered, and the obtained solutions depend on the choice of $f\,(R, \varphi)$ ETG model, since the geometric factor $F(x)$ is time dependent.*
- *Hence in this case, gravitomagnetic corrections to GR emerge with time-dependent sources*
- *The case of non-commutative spectral geometry that we discussed above deserves a final remark*
- *This model descends from a fundamental theory and can be considered as a particular case of ETGs*
- *Its parameters can be probed in the weak-field limit and at local scales, opening new perspectives worthy of further developments.*